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Some Remarks on the Algebra of Eddington's *E* Numbers

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This paper reviews the algebra of Eddington's E *numbers and identifies those points where Eddington anticipated results of current interest. He discovered the Majorana spinors, and was responsible for the standard* γ^5 *notation as well as the notion of chirality. Furthermore, Eddington defined Clifford algebras in eight and nine dimensions which are now appearing in grand unified gauge and supersymmetric theories. A point which Eddington cleared up, yet is still misunderstood, is that the Dirac algebra corresponds to a* five-*dimensional base space.*

With the occasion of the centennial of Sir Arthur Eddington's birth in 1982, Professor S. Chandrasekhar asked the author to review Eddington's algebraic work. The purpose of this was to clarify precisely in what way Eddington made mathematical advances with his theory of "*E* numbers." This question certainly arises every time one confronts Eddington's fascinating, puzzling, and controversial last works, the *Relativity Theory of Protons and Electrons*[1], and the *Fundamental Theory*.[2] The results of this study were reported at the Eddington Centenary Lectures in Cambridge,[3] and are presented here in detail.

Eddington apparently defined the real Majorana matrices (actually an equivalent set) before Majorana; he introduced the now standard γ^5 notation, and also the concept of chirality; he investigated the group structure of the Dirac algebra; and he defined the complex Clifford algebra over an eight-dimensional space which today appears in formulations of supersymmetry and supergravity.

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In addition to these main results, Eddington anticipated many of the formal connections between tensor geometry and Clifford algebras, although in a manner which was not entirely clear to his contemporaries. This note attempts to identify and clarify those particular features of his work which are of current interest, in the light of contemporary accounts of Clifford algebras.[4-11] Some previous surveys of Eddington's work may be found useful in this discussion,[12-14] and in following the original sources.[1,2]

Eddington used a set of anticommuting matrices, the "E symbols," which he introduced in Ref. 15, and which formed the framework of his later algebraic formulations.[1,2] These matrices have square equal to -1 and mutually anticommute. They therefore realize a Clifford algebra.[4-9] The E symbols can be given in terms of the familiar Pauli matrices τ_1, τ_2, τ_3 as follows:

$$E_1=\begin{pmatrix} i\tau_1 & 0 \\ 0 & i\tau_1 \end{pmatrix}, \quad E_2=\begin{pmatrix} i\tau_3 & 0 \\ 0 & i\tau_3 \end{pmatrix}, \quad E_3=\begin{pmatrix} 0 & -\tau_2 \\ \tau_2 & 0 \end{pmatrix}, \quad E_4=\begin{pmatrix} i\tau_2 & 0 \\ 0 & -i\tau_2 \end{pmatrix}$$

$$(E_\mu)^2=-1, \qquad \mu=1,2,3,4 \tag{1}$$

The Clifford algebra realized by these matrices is isomorphic to the algebra of the Dirac matrices. This is not, however, universally known. This is easy to show in two different ways. First, the E-symbols define a finite multiplicative group, which can be identified by determining its group order. The group elements are the 64 combinations listed below:

$$\begin{gathered}\{\pm 1,\ \pm E_\mu,\ \pm iE_\mu,\ \pm E_\mu E_\nu,\ \pm iE_\mu E_\nu, \\ \pm E_\mu E_\nu E_\lambda,\ \pm iE_\mu E_\nu E_\lambda,\ \pm E_1E_2E_3E_4,\ \pm iE_1E_2E_3E_4,\ \pm i\} \\ \mu,\nu,\lambda=1,2,3,4; \qquad \mu\neq\nu\neq\lambda\end{gathered} \tag{2}$$

We determine how many elements in (2) are of order two, i.e., have square equal to 1; and how many are of order four, i.e., have square equal to -1. The results are (counting the negative elements separately): 31 elements of order two; 32 elements of order four; the unit 1 is of order one. The group order is written as

$$\operatorname{order}(1,31,32)=1^1 2^{31} 4^{32} \tag{3}$$

The group order (3) identifies this finite group to be isomorphic to the group of the Dirac matrices.[8,10,11] Hence, the two algebras are isomorphic. (Note that, even though the group is of order 64, the algebra ignores the distinction between negative and positive elements, and is of order 32.)

Since Eddington gave the representation (1), we can provide a second, distinct proof of the equivalence of the algebra of E-numbers to the Dirac algebra. The representation space of the E-symbols is $\mathbb{C}(4)$ (4×4 complex), which is completely spanned. By universality, this Clifford algebra must be isomorphic to the Dirac algebra, which also completely spans $\mathbb{C}(4)$.[4-10] Therefore, the E-symbols (1) realize the Dirac algebra.

Eddington observed that the Dirac algebra in fact corresponds to a five-dimensional space over the real numbers.[4-10] We quote Eddington (Ref. 2, p. 125):

> "The E-frame provides a fifth direction perpendicular to the axes x_1, x_2, x_3, t."

Eddington gave the fifth basis element in the representation (1) as[15]

$$E_5 = \begin{pmatrix} 0 & i\tau_2 \\ i\tau_2 & 0 \end{pmatrix}, \qquad (E_5)^2 = -1 \tag{4}$$

Eddington apparently considered this fifth direction on an equal footing with the four space-time directions. He is not clear on any physical interpretation of this fifth direction, but this point of view is consistent with the geometrical picture of the Dirac algebra given in Refs. 8–11. I personally believe that it corresponds to the spin.

The Clifford algebra is an associative algebra, and therefore admits representations by matrices. In the mathematics literature, the different Clifford algebras are invariably denoted by the space of smallest irreducible matrix representations as $\mathbb{R}(n)$, $\mathbb{C}(n)$ and $H(n)$, [or $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, and $M_n(H)$], respectively.

In order to establish the different algebras of interest, we review those algebras which possess matrix representations within $\mathbb{C}(4)$. There are eleven distinct Clifford algebras which are representable in $\mathbb{C}(4)$ and its subspaces. In order to avoid possible confusion, it is convenient to employ the notation for Clifford algebras based on their correspondence to finite groups.[8-11]

The three largest of the algebras represented within $\mathbb{C}(4)$ are almost always confused.[10] The algebra S_2 completely spans the representation space $\mathbb{C}(4)$, and is realized by the Dirac and Eddington Matrices: S_2 corresponds to the Clifford algebras $A^{4,1} \approx A^{2,3} \approx A^{0,5}$. The other two algebras do *not* completely span $\mathbb{C}(4)$. One of them is the algebra N_3 realized by the real Majorana matrices: N_3 spans the $\mathbb{R}(4)$ subspace of $\mathbb{C}(4)$, and corresponds to the Clifford algebras $A^{3,1} \approx A^{2,2}$. The third algebra is N_4, which spans the $H(2)$ subspace of $\mathbb{C}(4)$, and corresponds to the Clifford algebras $A^{1,3} \approx A^{0,4} \approx A^{4,0}$ [including the Minkowski space (1, 3)]. The other subalgebras are of smaller dimension.[8,9]

It is not generally known that Eddington discovered a matrix representation of the Majorana algebra before Majorana. We recall this result from Ref. 16, p. 534.

"The distribution of the real and imaginary matrices in Dirac's identification seems very odd;$\cdots$ It would seem most natural to avoid introducing more imaginary coordinates than necessary by taking the identification

$$E_1=\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix},\quad E_2=\begin{pmatrix}\tau_3 & 0\\ 0 & -\tau_3\end{pmatrix},\quad E_3=\begin{pmatrix}\tau_1 & 0\\ 0 & -\tau_1\end{pmatrix},$$
$$E_4=\begin{pmatrix}-i\tau_2 & 0\\ 0 & i\tau_2\end{pmatrix},\quad E_5=\begin{pmatrix}0 & -\mathbb{1}\\ \mathbb{1} & 0\end{pmatrix} \tag{5}$$

...with our last identification, E_1, E_2, E_3, E_4 are real." ($\mathbb{1}$ is the 2×2 unit matrix.) The mutually anticommuting matrices (5) have squares equal to $+1, +1, +1, -1, -1$ times the 4×4 unit matrix, respectively, and provide a realization of the Clifford algebra N_3. By universality, this set is related to the matrices given by Majorana,[17] by a similarity transformation. Eddington does not appear to have used the matrices (5), since he adopted E-symbols with squares equal to -1 (1) in Ref. 15 for all his subsequent work[1,2] (and those realize not N_3, but S_2).

It is necessary for what follows to discuss the volume elements in the Clifford algebra. The volume elements in four and five dimensions can be defined in terms of an abstract algebraic basis as (here, the e_μ are not specific matrices)

$$\omega^4=e_1e_2e_3e_4,\qquad \omega^5=e_1e_2e_3e_4e_5 \tag{6}$$

The algebraic properties of the volume elements in any dimension are well-known consequences of anticommutation: ω^4 anticommutes with all the e_μ, $\mu=1, 2, 3, 4$, while ω^5 commutes with all the e_μ and with e_5. A five-dimensional Clifford algebra includes both ω^4 and ω^5, but a four-dimensional one includes only ω^4. The square of ω^5 in S_2 is -1.[8,9] By Schur's lemma, ω^5 can only be represented in $\mathbb{C}(4)$ by $\pm i$ times the 4×4 unit matrix. This is verified in the case of representation (1), (4) by computing the matrices corresponding to ω^4 and $\omega^5(6)$:

$$\omega^4\leftrightarrow\begin{pmatrix}0 & -\tau_2\\ -\tau_2 & 0\end{pmatrix},\qquad \omega^5\leftrightarrow-\begin{pmatrix}i\mathbb{1} & 0\\ 0 & i\mathbb{1}\end{pmatrix} \tag{7}$$

From definition (6), it follows that $\omega^4e_5=\omega^5$, which in the case of (7) gives, by inversion, $e_5=\omega^4\omega^5\leftrightarrow -i\omega^4$. [Compare (4) with (7).] The matrices representing e_5 and ω^4 in *any* representation of the Dirac algebra therefore differ by a factor of $\pm i$.[10]

In the distinct case of the real matrices (5), the four-dimensional volume element ω^4 is computed from (6) to be

$$\omega^4 \leftrightarrow \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{8}$$

This matrix is identical to the matrix labeled E_5 (5), even though there is no geometrical fifth dimension in the four-dimensional Majorana algebra. The above information is summarized in Table I.

The first extensive papers on the algebra of the Dirac matrices appeared in the same year.[16,18–20] None seems to be aware of Clifford's work. Both Tetrode and von Neumann identify the four-dimensional volume element $\gamma^1\gamma^2\gamma^3\gamma^4$, and show that it is a Lorentz invariant. They do not, however, explicitly show its anticommutation with the other γ^μ, nor do they label it as γ^5. Dirac does not explicitly identify γ^5. Eddington gave the five anticommuting matrices E_μ, E_5 in Ref. 16, and pursued a more general algebraic investigation in Ref. 15. A useful identification is to denote the volume element $\gamma^1\gamma^2\gamma^3\gamma^4$ by γ^5, following Eddington's work. It is not clear precisely who adopted this notation at first, but after Pauli's fundamental reviews,[21,22] it became standard practice.

Eddington used the term "chirality" to distinguish between right-handed and left-handed coordinate frames, which differ by a sign in the volume element ω^5 (the word was coined by Kelvin) (see Ref. 2, p. 111). When the chiral projection operators $\frac{1}{2}(1 \pm \gamma^5)$ were introduced in weak interactions by Watanabe, and Sudarshan and Marshak,[23,24] this term was adopted, even though Eddington had not used it in precisely the same way.

In his 1932 paper generalizing the Dirac matrices from flat space-time to a curved Riemannian space,[25] Schrödinger reviews the mathematics of the usual Dirac algebra, and includes other early references on the subject.

The important point in all of this is that the original confusion as to whether γ^5 is a volume element, or a geometrically relevant fifth generator, continues to this day. Eddington was apparently the only person to see this clearly. His contemporaries missed his geometrical explanation entirely, and, as a result, the physics literature perpetuates a mathematically ambiguous tradition. This is especially disturbing in the context of constructing algebraic models, where each dimension must be given a precise physical correspondence. In such cases, the role of the fifth dimension is problematical.

At this point we turn to a slightly different topic. In his investigations on the algebraic properties of the E-symbols, Eddington identified closed subsets of anticommuting combinations of E-symbols, labeling them "triads," "tetrads," and "pentads." He was in fact identifying the sub-

Table I. Matrix Representations of Clifford Algebras given by Eddington

	e_1	e_2	e_3	e_4	e_5	ω^4	ω^5
$A^{0,5}$ (Dirac)	$\begin{pmatrix} i\tau_1 & 0 \\ 0 & i\tau_1 \end{pmatrix}$	$\begin{pmatrix} i\tau_3 & 0 \\ 0 & i\tau_3 \end{pmatrix}$	$\begin{pmatrix} 0 & -\tau_2 \\ \tau_2 & 0 \end{pmatrix}$	$\begin{pmatrix} i\tau_2 & 0 \\ 0 & -i\tau_2 \end{pmatrix}$	$\begin{pmatrix} 0 & i\tau_2 \\ i\tau_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\tau_2 \\ -\tau_2 & 0 \end{pmatrix}$	$\begin{pmatrix} -i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix}$
$A^{3,1}$ (Majorana)	$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$	$\begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix}$	$\begin{pmatrix} \tau_1 & 0 \\ 0 & -\tau_1 \end{pmatrix}$	$\begin{pmatrix} -i\tau_2 & 0 \\ 0 & i\tau_2 \end{pmatrix}$	—	$\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$	—

algebras of S_2 such as the Lorentz Lie algebra $SO(1, 3)$ and the rotation Lie algebra $SO(3)$ in terms of a specific basis.

A close scrutiny of Eddington's "triads," "tetrads," and "pentads" of anticommuting elements[1,2] reveals a realization of some of these algebras. That was one of the first attempts at a systematic classification of the subalgebras of the Dirac algebra.

Eddington attached considerable importance to finding idempotent E-numbers, i.e., elements α of the algebra satisfying $\alpha^2 = \alpha$. We now know that the projection operators for momentum and spin in quantum electrodynamics are idempotent elements of the Clifford algebra. It is not clear, however, if Eddington ever made this identification. He does not appear to have used idempotents as projection operators in the usual manner.

Eddington went further to define the square of the E-algebra as 16×16 complex matrices. This "double E-frame" corresponds to one of the three Clifford algebras in a nine-dimensional base space, called S_4. S_4 is the complexification of the two algebras in eight dimensions, N_7 and N_8.[8,9] It turns out that much of the current work on supersymmetric gauge fields is based on eight- and nine-dimensional Clifford algebras. Eddington was in this respect very much ahead of his time.

One ought to stress, however, that Eddington did not clearly anticipate current physical supersymmetry theories. He did sense that a larger Clifford algebra would be useful in a symmetrical description of nature, and in this aspect he was entirely correct. The construction of the larger Clifford algebras is normally representation-specific, and follows, in a general manner, from the classic construction of Brauer and Weyl.[26] It is to Eddington's credit that his construction was largely representation-independent. We include a selection of references on the use of Clifford algebras in supersymmetry.[27-35]

In conclusion, this paper has reviewed Eddington's algebraic ideas with the intention of throwing light on some of his remarkable insights. The message which Eddington's work conveys to us is that the physical world has an intrinsic algebraic structure, yet the use of representation matrices frequently obscures or unnecessarily complicates this structure. For this reason, the formalism of E-numbers was originally developed to serve as a representation-free setting for the description of physical fields. Eddington's motivation for this is expressed in Ref. 1, p. 39:

> "I cannot believe that anything so ugly as the multiplication of matrices is an essential part of the scheme of nature."

We believe that this point of view anticipates the current representation-free formalisms of mathematical physics.

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