AN AMENDED MAGNETOHYDRODYNAMIC EQUATION WHICH PREDICTS FIELD-ALIGNED CURRENT SHEETS

N. A. SALINGAROS

Division of Mathematics, University of Texas, San Antonio, Texas, U.S.A.

(Received 22 May, 1987)

Abstract. A derivation of the physical conditions for magnetohydrodynamic equilibrium from first principles establishes a set of equations which differ slightly from the usual ones. The difference is only relevant in the special case when the current is parallel to the magnetic field. What is noteworthy is that these modified equations predict that the simplest magnetohydrodynamic equilibria can only exist in regions of field-aligned current sheets. This prediction is entirely consistent with the observed large-scale structures in the Earth's magnetosphere.

1. Introduction

The conditions for hydromagnetic equilibrium are of central importance in both plasma physics and in astrophysics. In particular, steady-state (often referred to as 'static') situations are of interest since it is precisely such phenomena which should be observed in nature. Physical candidates for hydromagnetic systems which are stable include (i) solar flares, (ii) field-aligned current sheets in the Earth's magnetosphere, and (iii) stable situations in confined thermonuclear plasmas. Much of the effort of theoretical plasma physics and astrophysics has been in obtaining mathematical solutions to the usual hydromagnetic equations, and then trying to correlate these solutions to observed phenomena.

This paper examines the physical conditions underlying hydromagnetic stability, starting from first principles. After a straightforward discussion we arrive at the hydromagnetic equations, but with a difference of two terms, one in the magnetic field and the other in the electric field. We discuss in which cases the extra magnetic term is unimportant, and analyze those cases where it may contribute to a divergence of the mathematical solution from physical reality. The modified equations predict the existence of hydromagnetic equilibrium when the total velocity-independent force vanishes. It is shown that this can only happen through the creation of large-scale field-aligned current sheets, with neutral sheets separating current sheets of opposite polarity. In contradistinction, the usual equations in this case predict configurations which do not correspond to observations of spatially resolved alternating current sheets. We are not aware of such a simple prediction of field-aligned current sheets in the literature.

Astrophysics and Space Science **137** (1987) 385–395.

2. One-Species Model

To begin, consider the elementary forces on charges in electromagnetic fields. Charged particles, whether static or moving in a current, experience the Lorentz force due to an external electromagnetic field. In addition, static charges generate their own electric field, and a current generates its own magnetic field via relativity. It is important to remember that there is no force due to the interaction of a particle with its own field, but only with external fields (Landau and Lifshitz, 1975; Jackson, 1975; Salingaros, 1986). It is true that a collection of particles will exert forces on each other, either electrostatic or magnetostatic, but all interparticle interactions will be ignored in this paper, as they complicate the second-order effects. All currents are due to the motion of charged particles: a discussion of the simple one-species charged gas is included here as a pre-requisite to describing the two-species neutral plasma.

Consider a gas of similarly charged particles with the same mass, charge density ρ^*, current density $\mathbf{j}$, which flows in a region of externally produced electric and magnetic fields $\mathbf{E}_0$ and $\mathbf{B}_0$. Denoting by primes the quantities in the particle's rest frame, the Lorentz force density on an element of the fluid (gas) is

$$\mathscr{F} = \rho^{*\prime}\mathbf{E}_0' . \tag{1}$$

An element of fluid will in general move with velocity $\mathbf{u}$ with respect to the laboratory frame.

Recall the Lorentz transformation laws for the electric and magnetic fields and for the charge-current four-vector (Landau and Lifshitz, 1975; Jackson, 1975). We ignore corrections of order $|\mathbf{u}|^2$ where $\mathbf{u}$ is the frame velocity, i.e., the Lorentz factor $\gamma = (1 - |\mathbf{u}|^2)^{-1/2}$ is taken to be approximately equal to one. We cannot, however, neglect the small-velocity relativistic effects described by the equations

$$\begin{cases} \mathbf{E}' \approx \mathbf{E} + \mathbf{u} \times \mathbf{B} , \\ \mathbf{B}' \approx \mathbf{B} - \mathbf{u} \times \mathbf{E} ; \end{cases} \tag{2a}$$

$$\begin{aligned} \mathbf{j}' &\approx \mathbf{j} - \rho^*\mathbf{u} , \\ \rho^{*\prime} &\approx \rho^* - \mathbf{u}\cdot\mathbf{j} . \end{aligned} \tag{2b}$$

The inverse transformations are achieved by switching primes and changing the sign of $\mathbf{u}$. In the laboratory frame, the Lorentz force density on an element of fluid is from (1) and (2a)

$$\mathscr{F} = \rho^{*\prime}(\mathbf{E}_0 + \mathbf{u} \times \mathbf{B}_0) . \tag{3}$$

Of course, the Lorentz force is intrinsically Lorentz covariant (Landau and Lifshitz, 1975; Jackson, 1975). A pragmatic choice of where to measure ρ^* is made: it is measured in the laboratory frame. The current defined by moving charges of charge density ρ^* is

$$\mathbf{j} = \rho^*\mathbf{u} , \tag{4}$$

which follows from (2b) since $\mathbf{j}' = \mathbf{0}$ in the particle's rest frame. With (4), one has

$\rho^{*\prime} \approx \rho^*(1 - |\mathbf{u}|^2) \approx \rho^*$, since relativistic corrections of order $|\mathbf{u}|^2$ are ignored. One arrives, therefore, at the usual Lorentz force law (Landau and Lifshitz, 1975; Jackson, 1975)

$$\mathscr{F} = \rho^* \mathbf{E}_0 + \mathbf{j} \times \mathbf{B}_0 \,. \tag{5}$$

Consider now the self-fields. A single charge q generates its own electric and magnetic fields in its rest frame: i.e.,

$$\mathbf{E}'_{\text{self}} = q \,\frac{\hat{\mathbf{r}}}{r^2}\,, \qquad \mathbf{B}'_{\text{self}} = \mathbf{0}\,. \tag{6}$$

Transforming to the laboratory frame one observes the self-fields as

$$\mathbf{E}_{\text{self}} \approx \mathbf{E}'_{\text{self}} \approx q \,\frac{\hat{\mathbf{r}}}{r^2}\,, \tag{7a}$$

$$\mathbf{B}_{\text{self}} \approx \mathbf{u} \times \mathbf{E}'_{\text{self}} = q\mathbf{u} \times \frac{\hat{\mathbf{r}}}{r^2}\,. \tag{7b}$$

The current density $\mathbf{j}$ is simply the integral over all moving charges, so that (7b) becomes the Biot–Savart law. The inverse of this is the Maxwell equation for a steady-state system (no displacement current: $\partial_t \mathbf{E} = \mathbf{0}$)

$$\mathbf{B}_{\text{self}} = \int \mathbf{j} \times \frac{\hat{\mathbf{r}}}{r^2}\, \mathrm{d}^3x \Leftrightarrow \nabla \times \mathbf{B}_{\text{self}} = \mathbf{j}\,. \tag{8}$$

A discussion of these basic and well-known facts is necessary in order to emphasize why a charge does not interact with its own self-fields (Salingaros, 1986). At the point $\mathbf{r} = \mathbf{0}$, for example, the electric self-field is singular, so that a charge is exactly on the singularity of its self-field. As for the magnetic self-field, that is zero in the charge's rest frame so the charge certainly experiences no force. Secondly, the assumption that there be no interactions between the individual particles (whether it be physically justified or not) rules out the consideration of a particle's interaction with other particles' self-fields, since that is equivalent to a particle-particle interaction. For this reason we have also ignored magnetization and electric polarization. The only interaction that is considered is between the fluid and the external fields.

The current $\mathbf{j}$ may be driven by sources outside the region of interest, or by forces of non-electromagnetic origin inside the region. In any case, the current interacts with $\mathbf{B}_0$ in a way so as to reduce the total magnetic field $\mathbf{B}_{\text{total}} = \mathbf{B}_0 + \mathbf{B}_{\text{self}}$. This process is easily seen in the steady-state situation of a current which is trapped in a homogeneous constant magnetic field $\mathbf{B}_0$. The charges rotate around the field lines with gyrofrequency $\mathbf{\Omega} = -q\mathbf{B}_0/m$ (Landau and Lifshitz, 1975; Jackson, 1975). Since the current is not driven, it assumes a circular configuration which generates a self-field antiparallel to $\mathbf{B}_0$. This phenomenon becomes very important later when stable steady-state configurations are described by self-fields which tend to be antiparallel with in order to minimize the external magnetic field. The total magnetic energy is thereby minimized.

At this point, we can write down the equations describing the simple one-species system. One has a charged gas where a fluid element moving with velocity $\mathbf{u}$ experiences the Lorentz force $\mathscr{F}$ in Equations (3) and (5). A steady-state situation presupposes no explicit time variation of the fields: $\partial_t \mathbf{E}_0 = \partial_t \mathbf{B}_0 = \mathbf{0}$. Also, since the current $\mathbf{j}$ is described in this model strictly by the relativistic motion of the charges (4), the external field $\mathbf{B}_0$ is a potential field. Otherwise, one would have to include an additional current $\mathbf{j}_0$ in the same region as $\mathbf{j}$, where $\mathbf{j}_0$ generates $\mathbf{B}_0$. The self-field (8) is time-independent because $\mathbf{j}$ is steady-state, i.e., $\mathbf{u}$ and, therefore, $\mathbf{j}$ are constant in time. The equations describing the system are:

$$\mathbf{E}_{\text{total}} = \mathbf{E}_0 = -\nabla\phi , \tag{9a}$$

$$\partial_t \mathbf{E}_0 = \nabla \times \mathbf{E}_0 = \mathbf{0} , \qquad \nabla \cdot \mathbf{E}_0 = -\nabla^2\phi = \rho^* , \tag{9b}$$

$$\mathbf{B}_{\text{total}} = \mathbf{B}_0 + \mathbf{B}_{\text{self}} = \mathbf{B}_0 + \int \mathbf{j} \times \frac{\hat{\mathbf{r}}}{r^2}\, \mathrm{d}^3x , \tag{9c}$$

$$\partial_t \mathbf{B}_{0,\,\text{self}} = \mathbf{0} , \quad \nabla \cdot \mathbf{B}_{0,\,\text{self}} = \mathbf{0} , \quad \nabla \times \mathbf{B}_0 = \mathbf{0} , \quad \nabla \times \mathbf{B}_{\text{self}} = \mathbf{j} , \tag{9d}$$

$$\Leftrightarrow \mathbf{B}_0 = \nabla\psi , \qquad \nabla^2\psi = 0 ; \qquad \nabla \cdot \mathbf{j} = 0 , \tag{9e}$$

$$\mathscr{F} = \rho^*\mathbf{E}_0 + \mathbf{j} \times \mathbf{B}_0 = -\rho^*\nabla\phi + (\nabla \times \mathbf{B}_{\text{self}}) \times \nabla\psi . \tag{9f}$$

From these equations one sees that $\mathbf{B}_0$ is a harmonic vector (Helgason, 1978, p. 144), whereas $\mathbf{B}_{\text{self}}$ contains no gradient part since it is entirely due to $\mathbf{j}$. It will be convenient from this point on to use the letter $\mathbf{h}$ for $\mathbf{B}_{\text{self}}$, so that the total magnetic field in this model always has a unique decomposition into a gradient and a term which contains no gradient (yet both terms can be written as the curl of vector potentials)

$$\mathbf{B}_{\text{total}} = \mathbf{B}_0 + \mathbf{B}_{\text{self}} = \nabla\psi + \mathbf{h} = \nabla \times \mathbf{A}_1 + \nabla \times \mathbf{A}_2 . \tag{10}$$

The hydromagnetic equation of motion is obtained by adding all possible forces on an element of the fluid. Since the velocity $\mathbf{u}$ can be space-dependent, we obtain a partial differential equation for $\mathbf{u}$ (Krall and Trivelpiece, 1973; Schmidt, 1979; Alexandrov *et al.*, 1984). Here, ρ_m is the mass density; p a scalar pressure (for simplicity); $\mathbf{g}$ the sum of linear external (non-electromagnetic) forces which may include the gravitational force; and ν the viscosity coefficient. The Lorentz force $\mathscr{F}$ is given by (9f) as

$$\rho_m(\partial_t + \mathbf{u} \cdot \nabla)\mathbf{u} = \mathscr{F} + \rho_m \mathbf{g} + \rho_m \nu \nabla^2 \mathbf{u} - \nabla p . \tag{11}$$

We are interested not in explicitly time-varying quantities, but in an equation for the steady-state fluid velocity $\mathbf{u}$. We accordingly take $\partial_t \mathbf{u} = \mathbf{0}$, and (11) becomes a second-order nonlinear inhomogeneous equation for $\mathbf{u}$. Using the identification $\mathbf{j} = \rho^*\mathbf{u}$, one has:

$$\mathbf{D}(\mathbf{u}) = \rho_m(\mathbf{u} \cdot \nabla - \nu\nabla^2)\mathbf{u} + \rho^*(\nabla\psi) \times \mathbf{u} = \rho^*\mathbf{E}_0 + \rho_m \mathbf{g} - \nabla p = \mathbf{Q}(\mathbf{E}_0, \mathbf{g}, p) . \tag{12}$$

Since this is a nonlinear system (because of the $(\mathbf{u} \cdot \nabla)\mathbf{u}$ term), we cannot simply add particular solutions, since their sum will not, in general, be a solution. One sees in the standard treatments that $\mathbf{u} = \mathbf{0}$ is the 'static' solution of the hydromagnetic equation (12). Nevertheless, $\mathbf{u} = \mathbf{0}$ is not a solution of (12) at all. It is a particular solution of the homogeneous equation $\mathbf{D}(\mathbf{u}) = \mathbf{0}$ corresponding to (12). As such, it is not helpful in obtaining a general solution of (12).

What the above step really signifies is the transition to an entirely different problem: the establishment of the conditions for electrohydrostatic equilibrium $\mathbf{Q} = \mathbf{0}$ in the particle's rest frame. One has a relationship between three independent forces (electrostatic, gravitational plus any other linear force, and the pressure). Knowledge of any two of them will provide the third from the condition $\mathbf{Q} = \mathbf{0}$. This problem does not involve either the fluid motion, or the magnetic field, but describes some large-scale static charge distribution. Physical situations likely to be representative of this condition are those where the interparticle interactions can be neglected, i.e., a rarefield electron gas in the Earth's magnetosphere. It is true that large-scale stable electrostatic configurations are observed there (see, for example, Alfvén, 1981).

There exists a justification for the above procedure, and it is a physical one. That is, 'the stability of a physical system should be Lorentz-invariant'. Therefore, one cannot expect equilibrium in a moving frame which is Equation (12) unless there is equilibrium in the rest frame of the fluid element $\mathbf{u} = \mathbf{0}$. We emphasize that this is an entirely independent condition which separates the problem into two parts: the homogeneous part of Equation (12), and Equation (12) in the rest frame of the fluid element.

$$\mathbf{D}(\mathbf{u}) = \mathbf{0}\,, \tag{13a}$$

$$\mathbf{Q} = \mathbf{0}\,. \tag{13b}$$

One still has to solve the homogeneous differential equation for $\mathbf{u}$ (13a), and knows that the solutions are physically relevant only in a situation given by the static balance Equation (13b). The above point, which is crucial, is not made explicit in the literature.

The conditions (13) are, in our opinion, the key to classical hydromagnetic equilibrium. From these follows the prediction of field-aligned current sheets presented later. For this reason, a separate later section is devoted to a more detailed derivation of Equation (13).

What is interesting in the charged gas model is the independence of the conditions of equilibrium on the magnetic field. We now proceed to discuss the two-species neutral plasma, where one can have genuine magnetohydrostatic equilibrium. The above description of the singly-charged fluid established the independent motions of the positive and negative charges in the external fields, but is rarely a realistic physical model by itself, because of the enormous interparticle electrostatic forces.

3. The Two-Species Model

Consider a fluid (plasma) containing oppositely charged particles, e.g., electrons and ions in equal numbers. In the rest frame of a fluid element, we take the electrons to have average velocity $\mathbf{v}_e$, and the ions to have average velocity $\mathbf{v}_i$. The charge densities are labelled ρ_e^* and ρ_i^*, respectively, and the plasma is assumed to be electrically neutral (Krall and Trivelpiece, 1973; Schmidt, 1979; Alexandrov *et al.*, 1984). A very careful analysis of the transformation properties of the current in different frames is necessary in order to recover a term in the electrostatic force that is usually neglected.

We have here four distinct Lorentz frames: the two rest frames of the two types of charges (denotes by double and triple primes); and the rest frame of the fluid element (denoted by a prime), which is moving with velocity $\mathbf{u}$ with respect to the (unprimed) laboratory frame. The velocity $\mathbf{u}$ is assumed to be non-relativistic, but no such assumption need hold true for the individual particle velocities $\mathbf{v}_e$ and $\mathbf{v}_i$. One has to be careful in deriving all the terms in the current by using the full Lorentz transformation rules (Landau and Lifshitz, 1975; Jackson, 1975; Salingaros, 1984), and not just (2b). With the current four-vector $j = (\mathbf{j}, j^4)$ proportional to the relativistic velocity $v = \gamma(\mathbf{v}, 1)$, $\gamma = (1 - |\mathbf{v}|^2)^{-1/2}$ (where e is the charge of a particle and n is the number density), one has:

$$j_e'' = -en(0, 1), \qquad j_i''' = en(0, 1),$$
$$j' = j_e' + j_i' = -en\gamma_e(\mathbf{v}_e, 1) + en\gamma_i(\mathbf{v}_i, 1) = en(\gamma_i\mathbf{v}_i - \gamma_e\mathbf{v}_e, \gamma_i - \gamma_e). \tag{14}$$

The transformation to the laboratory frame can be done by using the nonrelativistic approximation (2b), i.e., $|\mathbf{u}|^2 \approx 0$.

$$\begin{aligned} \mathbf{j} &\approx \mathbf{j}' + j^{4\prime}\mathbf{u} = en(\gamma_i(\mathbf{v}_i + \mathbf{u}) - \gamma_e(\mathbf{v}_e + \mathbf{u})), \\ j^4 &\approx j^{4\prime} + (\mathbf{u}\cdot\mathbf{j}') = en(\gamma_i(1 + \mathbf{u}\cdot\mathbf{v}_i) - \gamma_e(1 + \mathbf{u}\cdot\mathbf{v}_e)). \end{aligned} \tag{15}$$

The Lorentz force in the laboratory frame is given in terms of (15) as

$$\mathscr{F} = j^4\mathbf{E}_0 + \mathbf{j} \times \mathbf{B}_0. \tag{16}$$

What is not usually emphasized in the literature is that the electrostatic term is in general nonzero in a neutral plasma. Only when the two species travel in opposite and equal velocities in the rest frame of the fluid element $\mathbf{v}_e = -\mathbf{v}_i$ does j^4 vanish identically. (We wish to note especially that we refrain from using the usual identification of $\mathbf{u}$ as the center-of-mass velocity $\mathbf{u} = \Sigma m_\alpha n_\alpha \mathbf{v}_\alpha / \Sigma m_\alpha n_\alpha$, which for the simple case of electrons and ions in equal numbers is $\mathbf{u} \approx (m_e/m_i)\mathbf{v}_e + \mathbf{v}_i$, since that would remove the independent variable $\mathbf{u}$ from the equations.)

The equations for steady-state hydromagnetic equilibrium are, in the laboratory frame using (9), (11), (15), and (16), of the form

$$\rho_m(\mathbf{u}\cdot\nabla - \nu\nabla^2)\mathbf{u} - \rho_m\mathbf{g} + \nabla p = j^4\mathbf{E}_0 + \mathbf{j}\times\mathbf{B}_0 = j^4\mathbf{E}_0 + (\nabla\times\mathbf{h})\times\nabla\psi,$$
$$\nabla\cdot\mathbf{h} = \nabla^2\psi = 0. \tag{17}$$

We propose Equation (17) as the equation of state for an electrically neutral plasma in an electromagnetic field, neglecting all interparticle interactions. Using $\mathbf{j}'$ and $j^{4\prime}$ from (14), we can separate the magnetohydrodynamic Equation (17) into a canonical form similar to (12).

$$\begin{aligned}\mathbf{D}(\mathbf{u}) &= \rho_m(\mathbf{u}\cdot\nabla - \nu\nabla^2)\mathbf{u} + j^{4\prime}\mathbf{B}_0 \times \mathbf{u} - \mathbf{E}_0(\mathbf{j}'\cdot\mathbf{u}) \\ &= \mathbf{j}' \times \mathbf{B}_0 + j^{4\prime}\mathbf{E}_0 + \rho_m\mathbf{g} - \nabla p = \mathbf{Q}(\mathbf{B}_0, \mathbf{E}_0, \mathbf{g}, p)\,. \end{aligned} \tag{18}$$

As before, we will impose the invariance of equilibria under Lorentz transformations to obtain the following conditions for hydromagnetic stability:

$$\mathbf{D}(\mathbf{u}) = \mathbf{0}\,, \tag{19a}$$

$$\mathbf{Q} = \mathbf{0}\,. \tag{19b}$$

That is, systems with magnetohydrodynamic stability are described by a fluid velocity $\mathbf{u}$ which is a solution of Equation (19a), under the balance condition between external forces (19b).

We should emphasize that this is entirely distinct from what is usually stated in the literature, even though the basis for the derivation is the usual one. Condition (19b) may be investigated in special cases. First, $j^{4\prime}$ is small, and so this relativistic correction to the electrostatic force may be neglected. A problem of particular importance is the existence of magnetohydrodynamic equilibria when the pressure cancels the linear external forces (gravity included), i.e., $\rho_m\mathbf{g} = \nabla p$. In that case, there is only the magnetic force in $\mathbf{Q}$, which must vanish identically.

$$\mathbf{j}' \times \mathbf{B}_0 = 0\,. \tag{20}$$

A consequence of the assumption of zero electrostatic correction $j^{4\prime} \approx 0$ is that $\mathbf{j}' \approx \mathbf{j}$ in (15) and so one can write the laboratory measured current in (20). The physical result is the following: under the condition of balance between forces $\rho_m\mathbf{g} = \nabla p$, magnetohydrodynamic equilibrium can be achieved only when the current is parallel to the external magnetic field.

This is the well-known magnetic force-free model (Salingaros, 1986). Writing a scalar of proportionality β, one has, from (20) and $\mathbf{j}' \approx \mathbf{j}$:

$$\mathbf{j} = \beta\mathbf{B}_0\,. \tag{21}$$

We are, moreover, able to extend the usual result in two significant ways. (i) Our result is not restricted to magnetohydrostatic equilibria: it represents the physical conditions under which solutions of the general magnetohydrodynamic Equation (19a) will be stable. (ii) The physical conditions for equilibrium imply field-aligned current sheets without the necessity for any additional assumptions.

4. Prediction of Field-Aligned Current Sheets

To see this as a consequence of stability, recall the properties of the magnetic field in this model, Equations (9) and (10). The total observed magnetic field $\mathbf{B}_{\text{total}}$ can be uniquely decomposed (10) into a gradient part $\mathbf{B}_0 = \nabla\psi$ (9e) which is the external magnetic field, and a part $\mathbf{B}_{\text{self}} = \mathbf{h}$ (9d) which contains no gradient, since it is the self-field generated by the current $\mathbf{j} = \nabla \times \mathbf{h}$. The equilibrium equations are, therefore, from (21)

$$\nabla \times \mathbf{h} = \beta\nabla\psi\,, \tag{22a}$$

$$\nabla^2\psi = 0\,, \qquad \nabla\cdot\mathbf{h} = 0\,. \tag{22b}$$

First, we will demonstrate that β cannot be a constant in (21) and (22a). Take the divergence and curl of Equation (22a)

$$\nabla\psi\cdot\nabla\beta = 0\,, \qquad \nabla\psi \times \nabla\beta = \nabla^2\mathbf{h}\,. \tag{23}$$

We have assumed in the magnetic field decomposition (10) that $\mathbf{h}$ is not a harmonic vector $\nabla^2\mathbf{h} \neq \mathbf{0}$, otherwise $\mathbf{j} \equiv \mathbf{0}$. Therefore, one can rule out a simple proportionality in (21) and (22) with $\beta = \text{constant}$, since from (23) $\beta = \text{const.} \Leftrightarrow \beta \equiv 0$ in (22). Equation (23) says that $\beta(\mathbf{r})$ increases in a direction orthogonal to the field line of the external magnetic field $\mathbf{B}_0$, i.e., $\beta(\mathbf{r})$ is constant along any field line of $\mathbf{B}_0$. In fact, β is constant on a plane which contains the vectors $\mathbf{j} \parallel \mathbf{B}_0$ with the same $\beta(\mathbf{r})$. This establishes a lamination of the current. The coefficient β must change relatively rapidly. A reasonable model of an external gradient field $\mathbf{B}_0$ has a large-scale flow of field lines. The variation of $\beta(\mathbf{r})$ orthogonal to surfaces where β is the same defines the varying amplitude and direction of the current sheets via (21) and (22).

The simplicity of this derivation, and the natural condition implying that $\mathbf{j}$ has the same value on a surface containing magnetic field lines, has been a simple consequence of the magnetohydrodynamic equation (17). If we take as an example $\beta(\mathbf{r}) = \alpha \sin(\mathbf{k}\cdot\mathbf{r})$, $\alpha = \text{const.}$, one has field-aligned current sheets of alternating polarity separated by neutral sheets. The distance between adjacent current sheets of opposite direction is in this case equal to $\pi/|\mathbf{k}|$. We know from observation in rarefied plasma of the Earth's magnetosphere that field-aligned currents are primary features of large-scale stable situations (Alfvén, 1977; Stern, 1977, 1983; Saflekos *et al.*, 1982). The sheets themselves may drape in any configuration.

The experimental observation of field-aligned current sheets is deduced from measurements of $\mathbf{j}$ and $\mathbf{B}_{\text{total}}$ at a succession of points, collected by satellites in the Earth's magnetosphere (Saflekos *et al.*, 1982). A satellite passing through the center of a current sheet will measure only $\mathbf{B}_0$ there. This is obvious since the self-field $\mathbf{h}$ due to a current is, inside a conductor, proportional to the distance from the center of the conductor. For the simple model of current sheets of alternating polarity mentioned above, the measured magnetic field $\mathbf{B}_{\text{total}}$ will range from $\mathbf{B}_0$ in the interior of a current sheet to $\mathbf{B}_0 + 2\mathbf{h}$ on a neutral sheet between two current sheets of opposite polarity.

The theoretical model (21) is consistent with the usual assumption that the current is everywhere parallel to the total measured magnetic field (Alfvén, 1977; Stern, 1977, 1983). Where $\mathbf{j}$ is maximum we have $\mathbf{j} \parallel \mathbf{B}_0$, while $\mathbf{j}$ is minimum when $\mathbf{j} = 0$ and is trivially parallel to any vector, whether it is $\mathbf{B}_0$ (21) or $\mathbf{B}_0 + 2\mathbf{h}$ which is the total measured field. In the absence of this picture, however, the usual assumption can be misleading.

5. Intrinsically Relativistic, Non-Hydrodynamic Description of Plasma

The preceding discussion makes it clear that a plasma in an external field without any particle-particle interactions is an entirely relativistic problem. For this reason, we are led to reformulate the equilibrium equations in the simplest possible way by eliminating the hydrodynamic part altogether. Equilibrium will occur in the rest frame of the fluid element. One of the basic principles of relativity is that: 'two forces which compensate each other in one reference frame must do so in every other reference frame' (Pauli, 1958). The formulation of the problem is thereby considerably simplified.

The rest frame of the fluid element is characterized by zero velocity $\mathbf{u} = \mathbf{0}$. A steady-state situation requires that $\partial_t \mathbf{u} = \mathbf{0}$, so that the total derivative $d_t \mathbf{u} = \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}$ vanishes in the rest frame. The forces on each fluid element can be separated into the electromagnetic Lorentz force $\mathscr{F}$, and a non-electromagnetic linear force $\mathbf{f}$ which includes gravitation, pressure, and any other linear external force. Moreover, in most simple physical situations, this non-electromagnetic force could be derivable from a scalar potential Φ. This property is not, however, necessary for the equilibrium.

$$\mathbf{f} = \rho_m \mathbf{g} - \nabla p = \nabla \Phi . \tag{24}$$

The relativistic four-force corresponding to any vector force $\mathbf{f}$ is given in any reference frame moving with velocity $\mathbf{u}$ as ($\gamma = (1 - |\mathbf{u}|^2)^{-1/2}$) (Pauli, 1958):

$$f = \gamma(\mathbf{f}, \mathbf{f} \cdot \mathbf{u}) , \tag{25}$$

so that in the rest frame $f = (\mathbf{f}, 0)$.

There is no natural way in which to include a viscous force, but that is a manifestation of particle-particle interactions, and so should not be considered in this model here.

The equations for plasma equilibrium are easily written down in the rest frame of the fluid element (here denoted with a prime in keeping with the previous section). The current $j' = (\mathbf{j}', j^{4\prime})$ is given by Equation (14).

$$\mathscr{F}' + \mathbf{f}' = \mathbf{0} , \tag{26a}$$

$$\Rightarrow j^{4\prime} \mathbf{E}_0' + \mathbf{j}' \times \mathbf{B}_0' + \rho_m \mathbf{g} - \nabla p = \mathbf{0} . \tag{26b}$$

We have assumed that the fluid element at rest experiences the same gravitational and pressure forces as usual. This is precisely the balance condition obtained in the previous sections (Equations (18) and (19b)).

It should be noted that the absence of particle-particle interactions implies no viscosity as well as no resistance. Therefore, this model effectively possesses infinite conductivity in keeping with the traditional conception of a collisionless plasma (Krall and Trivelpiece, 1973; Schmidt, 1979; Alexandrov *et al.*, 1984).

We can give a picture of general steady-state situations which are governed by Equation (26). Using the approximation $j^{4\prime} \approx 0 \Rightarrow \mathbf{j}' \approx \mathbf{j}$ of the previous section, Equation (24) implies that the magnetic force is balanced by a total gradient,

$$\mathbf{j} \times \mathbf{B}_0 \approx -\nabla\Phi_{\text{total}} . \tag{27}$$

Now Φ_{total} includes the electrostatic potential (9a) as well as the pressure and gravitational potential (24). From Equation (27) follow the conditions for general steady-state magnetohydrodynamic equilibrium:

$$\mathbf{j} \cdot \nabla\Phi_{\text{total}} = 0 , \qquad \mathbf{B}_0 \cdot \nabla\Phi_{\text{total}} = 0 , \tag{28a}$$

$$(\mathbf{B}_0 \cdot \nabla)\mathbf{j} = (\mathbf{j} \cdot \nabla)\mathbf{B}_0 . \tag{28b}$$

Conditions (28a) imply that $\mathbf{j}$ and $\mathbf{B}_0$ must lie on the equipotential surfaces Φ_{total} = const. and be related by (28b) for equilibrium. In a general situation where the total gradient forces do not cancel, the current will transport charges across the magnetic field lines. When the total gradient forces cancel, one has the formation of field-aligned current sheets.

The above conditions (27) and (28) may seem very familiar (compare, for example, Krall and Trivelpiece, 1973, p. 98; Schmidt, 1979, p. 91 and exercise 4–1). In fact, they are entirely distinct from the usual ones, and coincide only in special cases.

6. Conclusion

This paper has derived conditions for magnetohydrodynamic equilibrium which are distinct from the usual ones. The main result which emerges even from a model which neglects all particle-particle interactions, is that stability is fundamentally an inhomogeneous property of the fluid. In the case of a charged gas, equilibrium is possible only with the formation of large-scale inhomogeneous charge distributions such as electrostatic double layers. For a neutral plasma, the conditions for magnetohydrodynamic stability imply the existence of field-aligned current sheets, with neutral sheets separating current sheets of opposite polarity.

This picture is drastically different from the usual association of stability with homogeneity, and, in particular, of instability with inhomogeneity. Nevertheless, we believe that the present model, although considerably simplified, is in basic agreement with experimentally observed situations in nature. The differences and similarities of the present theory with the usual magnetohydrodynamic theory can be critically discussed only within the context of the self-force, which is outside the scope of this paper.

References

Alexandrov, A. F., Bogdankevich, L. S., and Rukhadze, A. A.: 1984, *Principles of Plasma Electrodynamics*, Springer-Verlag, Berlin.

Alfvén, H.: 1977, *Rev. Geophys. Space Phys.* **15**, 271.

Alfvén, H.: 1981, *Cosmic Plasma*, D. Reidel Publ. Co., Dordrecht, Holland.

Helgason, S.: 1978, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York.
Jackson, J. D.: 1975, *Classical Electrodynamics*, 2nd ed., Wiley, New York.
Krall, N. A. and Trivelpiece, A. W.: 1973, *Principles of Plasma Physics*, McGraw-Hill, New York.
Landau, L. D. and Lifshitz, I. M.: 1975, *The Classical Theory of Fields*, 4th ed., Pergamon, Oxford.
Pauli, W.: 1958, *Theory of Relativity*, Pergamon, Oxford.
Saflekos, N. A., Sheehan, R. E., and Carovillano, R. L.: 1982, *Rev. Geophys. Space Phys.* **20**, 709.
Salingaros, N. A.: 1984, *J. Math. Phys.* **25**, 706.
Salingaros, N. A.: 1986, *J. Phys. A: Math. Gen.* **19**, L705.
Schmidt, A.: 1979, *Physics of High Temperature Plasmas*, 2nd ed., Academic Press, New York.
Stern, D. P.: 1977, *Rev. Geophys. Space Phys.* **15**, 156.
Stern, D. P.: 1983, *Rev. Geophys. Space Phys.* **21**, 125.